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# ELLIPSOIDAL ESTIMATES OF A CONTROLLED SYSTEM'S ATTAINABILITY DOMAIN*

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An inner ellipsoidal estimate (best, in a certain sense) of the attainability domain is obtained for a linear controlled system.

**1. Statement of the problem.** Consider the linear controlled system

$$dx/dt = C(t)x + K(t)u + f(t) \tag{1.1}$$

where $t$ is time, $x$ is the $n$-dimensional phase-coordinate vector, $u$ is the $m$-dimensional control vector, $C(t)$ and $K(t)$ are $(n \times n)$-and $(n \times m)$-matrices, respectively, and $f(t)$ is an $n$-dimensional vector. Functions $C(t)$, $K(t)$, $f(t)$ are continuous for $t \geqslant t_0$. The notation

$$x \in E(a, Q) \tag{1.2}$$

indicates that vector $x$ belongs to the ellipsoid

$$(Q^{-1}(x-a),\ x-a) \leqslant 1 \tag{1.3}$$

Here $a$ is the $n$-dimensional vector of the ellipsoid's center, $Q$ is a symmetric positive definite $(n \times n)$-matrix, and the parentheses denote the scalar product. As $Q \to 0$ the ellipsoid (1.2), (1.3) shrinks to the point $x = a$. Assume that the initial data and the controls for system (1.1) are ellipses

$$x(t_0) \in E(a_0, Q_0),\ u(t) \in E(0, G(t)),\ t \geqslant t_0 \tag{1.4}$$

Here $a_0$ is a specified $n$-dimensional vector and $Q_0$ and $G$ are symmetric positive definite $(n \times n)$- and $(m \times m)$-matrices, respectively.

The set of possible values of solutions $x(t)$ of system (1.1) at the instant $t$ for any $x(t_0)$ and $u(t)$ satisfying constraints (1.4) is called the attainability domain $M(t)$ for system (1.1), (1.4). It is well known that $M(t)$ is a bounded convex set. The attainability domain is an important characteristic of a controlled system /1— 3/ and is used when solving control theory and differential game problems. The effective construction of set $M(t)$ for sufficiently large $n$ is significantly difficult and requires a large amount of computations. Therefore, it is of interest to obtain simple estimates, both outer as well as inner, for set $M(t)$. In the present paper we examine ellipsoidal estimates of the form

$$E(a_-(t),\ Q_-(t)) \subset M(t) \subset E(a_+(t), Q_+(t)) \tag{1.5}$$

where $a_-$, $a_+$ are the centers and $Q_-$, $Q_+$ are the matrices of the ellipsoids; the subscript minus refers to the inner approximation and the plus, to the outer ones.

Certain outer ellipsoidal estimates of the attainability domain were constructed in /4— 6/. An outer ellipsoidal estimate, locally optimal in the following sense, was given in /7/. At each instant there is constructed an ellipsoid of least volume, described around the attainability domain and generated by the ellipsoidal approximation of this domain at a near preceding instant. This estimate is based on the optimal outer approximation of the sum of two ellipsoids (the ellipsoid at least volume, containing the sum of two ellipsoids, is constructed).

The functions $a_+$ and $Q_+$ for the outer estimate indicated satisfy the equation systems and initial conditions /7/

$$da_+/dt = C(t)a_+ + f(t),\ \ dQ_+/dt = C(t)Q_+ + Q_+C'(t) + qQ_+ + q^{-1}R(t) \tag{1.6}$$

$$R(t) = K(t)G(t)K'(t),\ \ q = \{n^{-1}\operatorname{Tr}[Q^{-1}R(t)]\}^{1/2},\ \ a_+(t_0) = a_0,\ \ Q_+(t_0) = Q_0$$

The primes denote transposition and Tr is the trace of the matrix. We obtain the outer estimate in (1.5) by integrating system (1.6).

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In the present paper we have constructed a locally-optimal inner estimate of the attainability domain, i.e., we have obtained equations for $a_-(t)$ and $Q_-(t)$ in (1.5). To do this we have solved, as a preliminary, an auxiliary problem on the optimal inner approximation of a sum of two ellipsoids: we have constructed the ellipsoid of greatest volume, contained in the sum of two ellipsoids.

2. **Auxiliary problem.** Let the $n$-dimensional vectors $x_1$ and $x_2$ belong to the ellipsoids

$$x_i \in E(a_i, Q_i), \quad i = 1, 2 \tag{2.1}$$

where $a_i$ are the vectors of the centers and $Q_i$ are symmetric positive definite matrices. If $x_1$ and $x_2$ take values from (2.1), then their sum lies in a closed convex region $S$, namely, the sum of the ellipsoids

$$x = x_1 + x_2 \in S = E(a_1, Q_1) + E(a_2, Q_2) \tag{2.2}$$

Let us construct the ellipsoid of greatest volume inscribed in $S$. At first we transform vectors $x_1$ and $x_2$ linearly, taking both ellipsoids (2.1) to the canonic form

$$y_i = A(x_i - a_i), \quad y_i \in E(0, D_i), \quad D_i = \operatorname{diag}(d_i^1, \ldots, d_i^n), \quad d_i^j > 0; \quad i = 1, 2; \quad j = 1, \ldots, n \tag{2.3}$$

The superscript $j$ denotes the element number. We note that under the linear transformation of vector $x$ the parameters of ellipsoid (1.2), (1.3) are transformed in the following manner /7/:

$$Ax + b \in E(Aa + b, \ AQA') \tag{2.4}$$

where $A$ is a matrix and $b$ is a vector. It is not difficult to verify formula (2.4) directly. According to (2.4), for transformation (2.3) we have

$$AQ_iA' = D_i, \quad i = 1, 2 \tag{2.5}$$

The transformation matrix $A$ taking ellipsoids (2.1) to canonic form is not unique. It can be found by solving the eigenvalue problem /8/

$$Q_1x = \lambda Q_2x, \quad \det(Q_1 - \lambda Q_2) = 0, \quad Q_1z_j = \lambda_jQ_2z_j, \quad \lambda_j > 0, \quad A' = \{z_1, \ldots, z_n\} \ (j = 1, \ldots, n) \tag{2.6}$$

Here the eigenvalues $\lambda_j$ are the roots of the characteristic equation and $z_j$ are the corresponding eigenvectors which serve as the columns of the transposed matrix $A'$. Among the roots $\lambda_j$ there can be multiple ones, but $n$ linearly-independent vectors $z_j$ always exist. We note that the elements of matrices (2.3) are expressed in terms of the eigenvalues $\lambda_j$ by the equations $d_1^j = \lambda_j$, $d_2^j = 1$, $j = 1, \ldots, n$.

We solve Eqs. (2.3) relative to $x_i$ and we represent sum (2.2) as

$$x = a_1 + a_2 + A^{-1}y, \quad y = y_1 + y_2 \in S_y = E(0, D_1) + E(0, D_2) \tag{2.7}$$

The semiaxes of ellipsoids $E(0, D_i)$ equal $(d_i^j)^{1/2}$. The convex region $S_y$ from (2.7) is contained in the parallelepiped

$$P: \ |y^j| \leqslant (d_1^j)^{1/2} + (d_2^j)^{1/2}, \quad j = 1, \ldots, n \tag{2.8}$$

Let us prove that the ellipsoid whose semiaxes equal the sums of the corresponding semiaxes of ellipsoids $E(0, D_i)$ is contained in $S_y$, i.e.,

$$E(0, D) \subset S_y \subset P, \quad D = \operatorname{diag}(d^1, \ldots, d^n), \quad d^j = [(d_1^j)^{1/2} + (d_2^j)^{1/2}]^2, \quad j = 1, \ldots, n \tag{2.9}$$

To prove this we take any point $y \in E(0, D)$ and set $y = y_1 + y_2$, where

$$y_i^j = (d_i^j/d^j)^{1/2}y^j, \quad i = 1, 2, \quad j = 1, \ldots, n \tag{2.10}$$

The superscript $j$ is the vector's component number. From the inclusion $y \in E(0, D)$, i.e., from the inequality

$$\sum_{j=1}^{n} (d^j)^{-1}(y^j)^2 \leqslant 1$$

follows, by virtue of relations (2.9) and (2.10)

$$\sum_{j=1}^{n} (d_i^j)^{-1}(y_i^j)^2 \leqslant 1, \quad i = 1, 2$$

Thus, any vector $y \in E(0, D)$ can be represented as a sum $y_1 + y_2$, where $y_i \in E(0, D_i)$. By the same token we have proved the inclusion $E(0, D) \subset S_y$.

Let us now prove that ellipsoid $E(0, D)$ has the greatest volume among all ellipsoids inscribed in $S_y$. By virtue of the inclusion $S_y \in P$ (see (2.9)), it is enough to prove that $E(0, D)$ has the greatest volume among all ellipsoids inscribed in $P$. By an extension of parallelepiped (2.8) along the coordinate axes, it is transformed into an $n$-dimensional cube $P_n^*$: $|y^j| \leqslant 1$, $j = 1, \ldots, n$, then ellipsoid (2.9) becomes the unit ball $|y| \leqslant 1$. It suffici to prove that the unit ball has the greatest volume among all ellipsods inscribed in cube $P_n^*$. This assertion is equivalent to the followires among all parallelepipeds circumscribed around the unit ball the cube $P_n^*$ has the least volume. The latter assertion can be proved by induction on the dimension $n$ of the space. The assertion is trivial for $n = 1$. For $n = 2$ it is obvious and signifies that the area of a square circumscribed around a circle is less than the area of any circumscribed parallelogram. Suppose that the assertion is true for some $n$. We consider an arbitrary $n + 1$-dimensional parallelepiped $P_{n+1}$ containing the $n + 1$-dimensional unit ball. The volume of $P_{n+1}$ equals the product of its height (it is not less than the ball's diameter) by the volume of the $n$-dimensional parallelepiped $P_n$ which is the base of $P_{n+1}$. The parallelepiped $P_n$ contains the projection of the $n + 1$-dimensional ball; consequently, $P_n$ contains the $n$-dimensional ball. By the assumption that the assertion is valid for $n$ the volume of $P_n$ is not less than the volume of cube $P_n^*$. Hence it follows that the volume of $P_{n+1}$ is not less than the volume of cube $P_{n+1}^*$. The assertion is proved.

Thus, the $E(0, D)$ from (2.9) is the ellipsoid of greatest volume inscribed in $S_y$. Returning to the variable $x$ in accord with relations (2.7) and (2.4), we obtain the desired ellipsoid of greatest volume inscribed in region $S$ of (2.2), in the form

$$E(a, Q) \subset S, \quad a = a_1 + a_2, \quad Q = A^{-1}D(A^{-1})', \quad D^{1/2} = D_1^{1/2} + D_2^{1/2}, \quad D_i = AQ_iA', \quad i = 1, 2 \tag{2.11}$$

To construct ellipsoid (2.11) we need to find the matrix $A$ of the transformation taking both the matrices $Q_1$ and $Q_2$ to diagonal form. For this it is sufficient to solve the eigenvalue problem (2.6).

3. Examples. 1°. Let ellipsods (2.1) be like and similarly oriented: $Q_1 = \nu Q_2$, where $\nu > 0$ is a scalar. Then, according to (2.11), we have

$$D_1 = AQ_1A' = \nu AQ_2A' = \nu D_2 \quad (Q_1 = \nu Q_2), \quad D = (\nu^{1/2} + 1)^2 D_2, \quad Q = (\nu^{1/2} + 1)^2 Q_2 \tag{3.1}$$

In this (and only in this) case the region $S$ is an ellipsoid and its approximation by the ellipsoid(2.11), (3.1) is exact. In particular, the approximation is exact in the one-dimensional case $(n = 1)$, as well as when ellipsoids (2.1) are balls.

2°. Let us consider the degenerate case when one of the ellipsoids $E(a_1, Q_1)$ is a segment of the $x^1$-axis, while the other is the unit ball

$$Q_1 = \operatorname{diag}(r^2, 0, \ldots, 0), \quad Q_2 = I \tag{3.2}$$

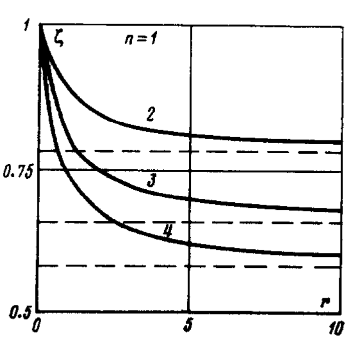


Fig.1

Here $r > 0$ is half the segment's length. Both ellipsoids (2.1) are in canonic form and we can take $A = I$. According to (2.11) we obtain

$$a = a_1 + a_2, \quad Q = \operatorname{diag}((r + 1)^2, 1, \ldots, 1) \tag{3.3}$$

The volume of the ellipsoid with parameters (3.3) equals

$$V_E = (r + 1)\, \pi^{n/2} [\Gamma(n/2 + 1)]^{-1} \tag{3.4}$$

where $\Gamma$ is the Euler gamma-function /9/. For comparison we present the volume of the region $S$, being the union of all unit balls with centers on the segment $a_1 + a_2 + e^1x^1$, where $e^1$ is the unit vector on the $x^1$-axis, $|x^1| \leqslant r$. Region $S$ consists of a cylinder of height $2r$ with a base in the form of a $n - 1$-dimensional unit ball and of two $n$-dimensional unit hemispheres with centers at the segment's endpoints. The volume of region $S$ equals

$$V_S = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} + \frac{2r\pi^{(n-1)/2}}{\Gamma[(n + 1)/2]} \tag{3.5}$$

We set up the ratio of volumes (3.4) and (3.5), characterizing the approximation accuracy

$$\zeta(r, n) = \frac{V_E}{V_S} = (r+1)\left\{1 + \frac{2r\Gamma(n/2+1)}{\pi^{1/2}\,\Gamma[(n+1)/2]}\right\}^{-1} \tag{3.6}$$

Using the formulas for the gamma-function /9/, from (3.6) we obtain the special cases

$$\zeta(r, n) = 1 - r\left[\frac{2\Gamma(n/2+1)}{\pi^{1/2}\,\Gamma[(n+1)/2]} - 1\right], \quad r \to 0$$

$$\zeta(\infty, n) = \frac{\pi^{1/2}\Gamma[(n+1)/2]}{2\Gamma(n/2+1)}, \quad \zeta(\infty, 2) = \pi/4, \quad \zeta(\infty, 3) = 2/3, \; \zeta(r, n) = (1 + r^{-1})\,\pi^{1/2}(2n)^{-1/2} \quad (n \to \infty)$$

Functions (3.6) are given in Fig.1.

**4. Estimate of the attainability domain.** We go on to derive the equations for $a_-(t)$ and $Q_-(t)$ from estimate (1.5). We prescribe a sufficiently small increment $h$ and we write the finite-difference approximation of system (1.1)

$$x(t+h) = x_1 + x_2, \quad x_1 = (I + hC)x + hf, \quad x_2 = hKu \tag{4.1}$$

Here the argument $t$ has been omitted. Suppose that the left-hand inclusion in (1.5) holds at instant $t$. Then from the constraint in (1.4) on the control and from (4.1) and (2.4) we obtain

$$x_1 \in M_1, \; M_1 \supset E(a_- + h(Ca_- + f), \; (I + hC)Q_-(I + hC)'), \; x_2 \in E(0, h^2R), \; R = KGK' \tag{4.2}$$

where $M_1$ is a set containing ellipsoid (4.2). We define a nonsingular matrix $A$ by the relations

$$AQ_-A' = D_1, \quad ARA' = D_2, \quad R = KGK' \tag{4.3}$$

where $D_1$ and $D_2$ are some diagonal matrices. Equalities (4.3) signify that the transformation with matrix $A$ leads (to within terms of higher order in $h$) both ellipsoids (4.2) to canonic form. In order to obtain $a_-(t+h)$ and $Q_-(t+h)$ we apply to ellipsoids (4.2) the formula (2.11) defining ellipsoid inscribed in region $S$. We obtain

$$a_-(t+h) = a_- + h(Ca_- + f), \; Q_-(t+h) = A^{-1}\{[A(I + hC)Q_-(I + hC)'A']^{1/2} + h(ARA')^{1/2}\}^2(A^{-1})' \tag{4.4}$$

We transform the formula in (4.4) for $Q_-(t+h)$ by expanding it in a series in $h$ and dropping the terms $o(h)$

$$Q_-(t+h) = A^{-1}\{[AQ_-A' + h(CQ_- + Q_-C')]^{1/2} + h(ARA')^{1/2}\}^2(A^{-1})' = A^{-1}\{AQ_-A' + h[CQ_- + Q_-C' + 2(AQ_-A')^{1/2}(ARA')^{1/2}]\}(A^{-1})' \tag{4.5}$$

Dividing the equality in (4.4) for $a_-(t+h)$ and the equality (4.5) for $Q_-(t+h)$ by $h$ and passing to the limit as $h \to 0$, we obtain the systems of differential equations

$$da_-/dt = C(t)a_- + f(t) \quad dQ_-/dt = C(t)Q_- + Q_-C'(t) + 2A^{-1}(AQ_-A')^{1/2}(AR(t)A')^{1/2}(A^{-1})' \tag{4.6}$$
$$R(t) = K(t)G(t)K'(t)$$

The initial conditions for systems (4.6) are obtained from (1.4)

$$a_-(t_0) = a_0, \quad Q_-(t_0) = Q_0 \tag{4.7}$$

The determination of the required parameters of ellipsoid $E(a_-(t), Q_-(t))$ has been reduced to the solving of the Cauchy problems (4.6), (4.7). The vector $a_-(t)$ of the ellipsoid's center satisfies the linear matrix system in (4.6), while the symmetric positive definite matrix $Q_-(t)$ satisfies the nonlinear matrix system in (4.6). The matrix $A$ occurring in the latter system depends on $t$ and $Q_-(t)$ and must satisfy equalities (4.3); its determination reduces to solving for each $t$ an eigenvalue problem of type (2.6). The matrices $(AQ_-A')^{1/2}$ and $(ARA')^{1/2}$ in (4.6) are diagonal according to (4.3) and, therefore, commute. The systems in (4.6) for $a_-$ and $Q_-$ are independent and can be itegrated individually. The system in (4.6) and the initial condition in (4.7) for $a_-(t)$ coincide with (1.6) for $a_+(t)$, and $a_-(t) \equiv a_+(t)$. The systems in (4.6) and (1.6) for $Q_-$ and $Q_+$ are similar in structire; see /7,10/. Solving the Cauchy problems (4.6), (4.7), we obtain the desired inner ellipsoidal estimate (1.5) of the attainability domain.

We note that ellipsoid $E(a_-, Q_-)$ is not the ellipsoid of greatest volume contained in the attainability domain $M$; in the construction of this ellipsoid the maximization of the volume held only locally, "in the small". Estimates (1.5), (4.6), (4.7) can be used only when constraints (1.4) are nonellipsoidal. In this case we should first construct ellipsoids contained in the sets of initial data and constraints on the control, and, next, apply the results obtained. Estimates (1.5) can be used for estimating the attainability domains of controlled systems, for estimating the perturbations (if $u(t)$ is a perturbation), for obtaining guaranteed estimates in differential games on the basis of the extremal aiming rule /1, 2/. Thus, if the inner estimate of the pursuer's attainability domain absorbs the outer estimate of the evader's attainability domain, then absorption automatically obtains for the attainability domains, which under the conditions specified guarantees capture. Estimates (1.5) enable us also to obtain two-sided bounds on the functional's extremum in optimal control problems. For a terminal functional of form $J = F(x(T))$, where $F(x)$ is a prescribed scalar function and $T > t_0$ is the fixed process termination instant, we have the obvious two-sided estimates

$$\min_{x\in E_+} F(x) \leqslant \min J = \min_{x\in M(T)} F(x) \leqslant \min_{x\in E_-} F(x), \quad E_- = E(a_-(T), Q_-(T)),\ E_+ = E(a_+(T), Q_+(T))$$

5. As an example we consider the second-order controlled system (the subscripts denote components of vectors $x$ and $a_-$)

$$dx_1/dt = x_2,\ dx_2/dt = u, \quad |u| \leqslant 1,\ n = 2,\ m = 1 \tag{5.1}$$

For system (5.1) the matrices and vectors in (1.1) and (4.6) equal

$$C = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad K = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad f = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}, G = 1, \quad R = KGK' = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix} \tag{5.2}$$

We take matrix $A$ in the form ($Q_{ij}$ are the elements of matrix $Q_-$)

$$A = \begin{Vmatrix} 1 & 0 \\ -Q_{12} & Q_{11} \end{Vmatrix} \tag{5.3}$$

It can be verified that matrix (5.3) satisfies conditions (4.3) if $R$ is given by formula (5.2). Substituting equalities (5.2) and (5.3) into system (4.6), we obtain

$$da_1/dt = a_2,\quad da_2/dt = 0,\quad dQ_{11}/dt = 2Q_{12},\quad dQ_{12}/dt = Q_{22},\quad dQ_{22}/dt = 2(Q_{11}Q_{22} - Q_{12}^2)^{1/2}Q_{11}^{-1/2} \tag{5.4}$$

For simplicity we assume the initial ellipsoid (1.4) as a point, namely, the origin, so that the initial conditions (4.7) have the form

$$a_i(0) = Q_{ij}(0) = 0, \quad i, j = 1, 2 \tag{5.5}$$

The solution of system (5.4) for $a(t)$ with initial conditions (5.5) is $a(t) \equiv 0$. We seek the solution of Cauchy problem (5.4), (5.5) for $Q_-(t)$ in the form

$$Q_{11} = b_1t^4, \quad Q_{12} = b_2t^3, \quad Q_{22} = b_3t^2 \tag{5.6}$$

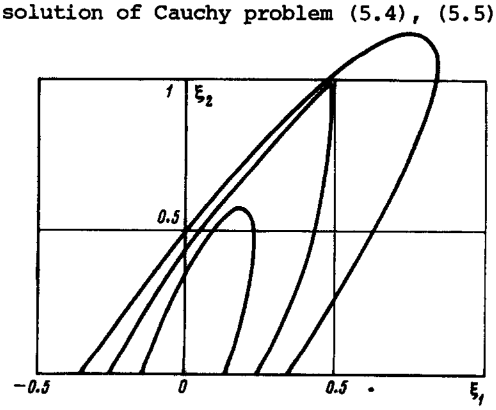


Fig.2

where $b_1, b_2, b_3$ are undetermined coefficients. Substituting (5.6) into (5.4), we find the unique nontrivial solution

$$b_1 = 1/18, \quad b_2 = 1/9, \quad b_3 = 1/3 \tag{5.7}$$

We introduce the variables

$$\xi_1 = x_1t^{-2}, \quad \xi_2 = x_2t^{-1} \tag{5.8}$$

In variables (5.8) the ellipsoid (1.3) with center $a(t) = 0$ and with matrix (5.6), (5.7) becomes

$$E(a_-, Q_-):\ 54\xi_1^2 - 36\xi_1\xi_2 + 9\xi_2^2 \leqslant 1 \tag{5.9}$$

The semiaxes $c_{1,2}$ of ellipse (5.9) and the angle of inclination $\alpha$ of its major semiaxis with the $x_1$-axis equal

$$c_{1,2} = [7 \pm (41)^{1/2}]^{1/2}/6,\ c_1 = 0{,}6102, \quad c_2 = 0{,}1288,\ \alpha = \operatorname{arctg}\{[(41)^{1/2} + 5]/4\} = 70°, 67$$

Fig.2 shows, in variables (5.8) (only the region $\xi_2 \geqslant 0$ is shown), the ellipse $E(a_-, Q_-)$, the exact attainability domain $M$ of system (5.1) (it is bounded by arcs of two parabolas), as well as the ellipse $E(a_+, Q_+)$ constructed by integrating system (1.6). In variables (5.8) its equation is (see /10/)

$$E(a_+, Q_+) : (135/16)\, \xi_1^2 - (45/4)\, \xi_1 \xi_2 + (9/2)\, \xi_2^2 \leqslant 1$$

The areas of the three domains indicated equal, respectively,

$$V_- = \pi / (9 \cdot 2^{1/2}) = 0{,}2468, \quad V_M = 2/3, \quad V_+ = 8\pi / (9 \cdot 5^{1/2}) = 1{,}2489$$

**6. Nonlinear systems.** The results obtained above for linear systems (1.1) can be extended to the following class of nonlinear systems:

$$dx/dt = C(t)x + v(x, u, t) \tag{6.1}$$

Here $v$ is a prescribed nonlinear vector-function of its arguments; the remaining notation is the same as in (1.1). Relative to function $v(x, u, t)$ we assume that the set $W(x, t)$ of its values (for any fixed $x$, $t \geqslant t_0$, and for all possible admissible controls $u$) satisfies the relations

$$E(f_-(t),\ G_-(t)) \subset W(x, t) \subset E(f_+(t),\ G_+(t)) \tag{6.2}$$

Here $f_-$ and $f_+$ are $n$-dimensional vectors and $G_-$ and $G_+$ are symmetric $(n \times n)$-matrices. These vectors and matrices are prescribed as time functions for $t \geqslant t_0$ and define the ellipsoids (6.2). Comparing relations (1.1), (1.4) and (6.1) and (6.2), we arrive at the following conclusion. All the estimates presented above are valid as well for the nonlinear systems (6.1) if everywhere in these estimates we set $K(t) = I$ and, in addition, we replace $f$ and $G$ by $f_-$ and $G_-$ (for the inner estimates, in formulas (4.1)—(4.7)) or by $f_+$ and $G_+$ (for the outer estimates, in formulas (1.6)). The estimates thus obtained are applicable for a rather wide class of nonlinear systems (6.1), (6.2), whose right-hand sides can be represented as a sum of a linear part not containing the control and of nonlinear bounded summands. These estimates can be improved if we restrict ourselves to a narrower class of nonlinearities, making the form of function $v$ concrete.

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